1. Plancherel algeba

① inner Hom

G reductive group, X spherical variety.

$\mathrm{Sat}_G := D(G_O \backslash G_F / G_O)$ acts on $D(X_F/G_O)$ via convolution,

$\delta_X := IC_0 \in D(X_F/G_O)$ is the push forward of constant sheaf on X_O/G_O.

we can define inner Hom, $\underline{\mathrm{End}}(\delta_X) \in \mathrm{Sat}_G$ represented by

$$\mathrm{Hom}_{\mathrm{Sat}_G}(S, \underline{\mathrm{End}}(\delta_X)) = \mathrm{Hom}_{X_F/G_O}(\delta_X * S, \delta_X).$$

This equips with canonical map $\delta_X * \underline{\mathrm{End}}(\delta_X) \xrightarrow{\mathrm{act}} \delta_X$.

composition $\delta_X * \underline{\mathrm{End}}(\delta_X) * \underline{\mathrm{End}}(\delta_X) \to \delta_X * \underline{\mathrm{End}}(\delta_X) \to \delta_X$

gives $\underline{\mathrm{End}}(\delta_X) * \underline{\mathrm{End}}(\delta_X) \to \underline{\mathrm{End}}(\delta_X)$,

together with unit $\delta_G \to \underline{\mathrm{End}}(\delta_X)$ gives an algebra structure.

Commutativity can be obtained from the compatibility with factorization structure.

② deequivariantization

$\mathcal{C}$ is $\mathrm{Rep}(G)$ linear, we can define $\mathcal{C}_{\mathrm{deeq}}$ as

objects is the same, $\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, \mathcal{O}(G) \otimes Y)$.

The composition is given by ring structure of $\mathcal{O}(G)$.

If $\mathcal{C} = \mathrm{Coh}(M/G)$, then $\mathcal{C}_{\mathrm{deeq}} = \mathrm{Coh}(M)$, and the functor $\mathcal{C} \to \mathcal{C}_{\mathrm{deeq}}$ by pullback.

In the case of $\mathcal{C} = \mathrm{Sat}_G = \mathrm{Coh}(\check{\mathfrak{g}}^*[2]/G^\vee)$,

we have $PL_X := \mathrm{Hom}_{\mathrm{Sat}_{G,\mathrm{deeq}}}(\mathcal{O}, \underline{\mathrm{End}}(\delta_X)) = \mathrm{Hom}_{\mathrm{Sat}_G}(\bigoplus_\lambda (V_\lambda \otimes \mathcal{O}) \otimes V_\lambda^*, \underline{\mathrm{End}}(\delta_X))$

$$= \bigoplus_\lambda \mathrm{Hom}_{X_F/G_O}((\delta_X * IC_\lambda) \otimes V_\lambda^*, \delta_X) = \bigoplus_\lambda \mathrm{Hom}_{X_F/G_O}(\delta_X, (\delta_X * IC_\lambda)) \otimes V_\lambda^*.$$

This corresponds to $\mathcal{O}(\check{M})$ in relative Langlands

③ cohomology

In the case X is a vector space,

as $\mathrm{Hom}_{\mathrm{Sat}_G}(\underline{\mathbb{C}}, \underline{\mathrm{End}}(\delta_X)) = \mathrm{Hom}_{X_F/G_O}(\delta_X * \underline{\mathbb{C}}, \delta_X) = H_*^{G_O}(R)$

where $R = \{(g, x) : g \in Gr_G,\ x \in X_O,\ gx \in X_O\}$,

the cohomology is exactly the algebra of Coulomb branch.

In relative Langlands, this should be $\check{M}/G^\vee \times_{\check{\mathfrak{g}}^*/G^\vee} \Sigma_{G^\vee}$.

2. Coulomb branch no spherical requirement

G group, N a representation $\mathcal{M}_C(G, T^*N) = \operatorname{Spec} H_*^{G_O}(R)$.

① abelian examples $G = \mathbb{G}_m$, N 1-dim rep of weight n.

$$R = \bigsqcup_{k\in\mathbb{Z}} \{t^k\} \times (N_O \cap t^k \cdot N_O).$$

As $H^*_{G\times\mathbb{G}_m^{rot}}(pt) = \mathbb{C}[a,\hbar]$-mod, it has generators $x = [N_O \cap tN_O]$, $y = [N_O \cap t^{-1}N_O]$.

one can calculate $x * y = e(N_O / t\cdot N_O) = (a+\frac{1}{2}\hbar)(a+\frac{3}{2}\hbar)\cdots(a+(n-\frac{1}{2})\hbar)$,

$y * x = e(t^{-1}N_O/N_O) = (a-\frac{1}{2}\hbar)\cdots(a-(n-\frac{1}{2})\hbar)$.

by putting $\mathbb{G}_m^{rot}$ acts via $-\frac{1}{2}$ on N, this quantized algebra only depend on $T^*N = N\oplus N^*$.

Other cases can be computed similarily:

$N =$ n-dim rep of weight 1, $x*y = (a+\frac{1}{2}\hbar)^n$, $y*x = (a-\frac{1}{2}\hbar)^n$.

② localization T acts on R, and $R_{G,N}^T = R_{T,N}$.

as $H_G^*(pt)$-mod, we have canonical map $\mathcal{M}_C \to t/W$.

the inclusion $R \hookrightarrow G_K \times_{G_O} N_O$ gives the homomorphism

$$z^* : H_*^{G_O}(R) \to H_*^{G_O}(G_K \times_{G_O} N_O) = H_*^{G_O}(Gr).$$

Prop: this is an algebraic hom, the functorality only happens for $N \hookrightarrow N\oplus M$.

by Localization theorem, they are isomorphic over t°:

$$H_*^{G_O}(R) = (H_*^{T_O}(R))^W \to H_*^{T_O}(R) \to H_*^{T_O}(R)|_{t^\circ} \xleftarrow[\varphi]{\sim} H_*^{T_O}(R_{T,N})|_{t^\circ},$$

To calculate multiplication, can further embed into $H_*^{T_O}(Gr_T)$.

given by inclusion $R_{T,N} = R_{G,N}^T \hookrightarrow R_{G,N}$

$H_*^{T_O}(R_{T,N})^W \to H_*^{G_O}(R)$

This gives the birational map

$$\begin{tikzcd} (t\times T^\vee)/W \arrow[r, dashed] \arrow[dr] & \mathcal{M}_C \arrow[d] \\ & t/W \end{tikzcd}$$

in particular $\dim \mathcal{M}_C = 2\operatorname{rk} G$.

③ rank 1 example

Prop $\mathcal{M}_C(G,N)|_t = \mathcal{M}_C(Z_G(t), N^t)$, thus for $t \in t\backslash t^\circ$, codim 1, suffice G rank 1.

$G = PGL_2$, $H_G^*(pt) = \mathbb{C}[t]^{\mathfrak{S}_2} = \mathbb{C}[\delta = t^2]$, $\eta = [R_{\frac{1}{2}}]$, $\xi = t\eta$.

a grading by stratification of Gr_G $\Rightarrow$ $\xi^2 - \delta\eta^2 \in \mathbb{C}[\delta]$.

the precise result, depending on N, can be calculated via the above embedding

$$\xi^2 - \delta\eta^2 = 4 \prod_{\mu \text{ wt of } N} (\mu t)^{|\mu|}.$$

$N = ad$, weights $-1, 0, 1$, $RHS = 4t^2 = 4\delta$.

The algebra $\mathbb{C}[\delta,\xi,\eta]/(\xi^2-\delta\eta^2-4\delta)$ is isomorphic to $\mathbb{C}[t,x,x^{-1}]^{\mathfrak{S}_2}$.

Cor. When N is the adjoint rep, we always have $\mathcal{M}_C = (t\times T^\vee)/W$.

3. Cherkis bow variety

① definition affine type A

building blocks $\overset{m}{-}\!\circ\!\overset{n}{-}$ $T^*\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ with $GL_m \times GL_n$ action

$\overset{m}{-}\!\times\!\overset{n}{-}$ $\begin{cases} T^*GL_n \times \mathbb{C}^n \times (\mathbb{C}^n)^* & GL_n \times GL_n \text{ acts on } GL_n,\ GL_n \text{ acts on } \mathfrak{gl}_n \times \mathbb{C}^n \times \mathbb{C}^n. \\ GL_n \times S(1^m, n-m) & GL_n \text{ acts on } GL_n,\ GL_m \text{ acts on } GL_n \times S. \end{cases}$

by choosing a direction, $\mathcal{M}_{bow} = \prod -\!\otimes\!- /\!/\!/ GL(n)$.

Note: for a slightly symmetric def of $\overset{m}{-}\!\times\!\overset{n}{-}$, it is by quiver
$$\begin{tikzcd} & \mathbb{C} \arrow[dr] & \\ \mathbb{C}^m \arrow[ur] \arrow[rr] \arrow[loop below] & & \mathbb{C}^n \arrow[loop below] \end{tikzcd}$$
with stability condition.

balanced: for all $\overset{m}{-}\!\circ\!\overset{n}{-}$, $m=n$. cobalanced: for all $\overset{m}{-}\!\times\!\overset{n}{-}$ $m=n$.

② theorems

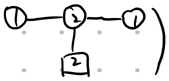
for cobalanced, it is isomorphic to quiver variety (Higgs branch)

for balanced, it is isomorphic to Coulomb branch of cor. quiver.

Proof
$$\begin{tikzcd} \mathcal{M}_{bow} \arrow[rr, dashed] \arrow[dr, "x"'] & & \mathcal{M}_C \arrow[dl] \\ & A^{n} & \end{tikzcd}$$
only need to show in the case of rank one.

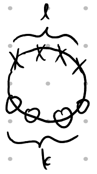
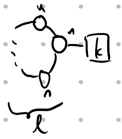
③ Hanany-Witten transition

$\overset{v_1}{-}\!\times\!\overset{v_2}{-}\!\circ\!\overset{v_3}{-} = \overset{v_1}{-}\!\circ\!\overset{v_2'}{-}\!\times\!\overset{v_3}{-}$, $v_2' = v_1 + v_3 - v_2 + 1$.

Example 1: $\mathcal{M}_H($  $) = \mathcal{M}_{bow}(-\circ\overset{1}{-}\circ\overset{2}{-}\times\overset{2}{-}\times\overset{2}{-}\circ\overset{1}{-}\circ-)$

$= \mathcal{M}_{bow}(-\times\overset{2}{-}\circ\overset{2}{-}\circ\overset{2}{-}\circ\overset{2}{-}\circ\overset{2}{-}\times-) = \mathcal{M}_C($  $)$

Comes from D3 branes in IIB string theory

Example 2: $\mathcal{M}_H($  $) = \mathcal{M}_{bow}($  $) = \mathcal{M}_C($  $)$.

Comes from $\overset{11\text{-dim'l}}{M}$ theory on $\mathbb{R}^3 \times \mathbb{C}^2/K_1 \times \mathbb{C}^2/K_2$ $K_1, K_2 \subset SU(2)$ discrete

with n M2 branes on $\mathbb{R}^3 \times 0 \times 0$

3d MS $\mathcal{M}_H = \{ n\ ADE(K_1) \text{ instanions on } \mathbb{C}^2/K_2 \}$

$\mathcal{M}_C = \{ n\ ADE(K_2) \text{ instanions on } \mathbb{C}^2/K_1 \}$